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Positive-Operator-Valued Measures and Projection-Valued Measures of Noncommutative Time Operators

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Some relationships between two different concepts of noncommutative time operators are discussed. One is the concept of a Hermitian, but not self-adjoint time operator T_B based on a positive-operator-valued measure for a dynamical observable B. The other is the concept of a self-adjoint time operator T_L obtained in the Liouville representation, a special case of the standard representation of quantum theory. Conditions are indicated under which a self-adjoint extension of T_B leading to T_L can be constructed. Similarities with the notions of consistent and inconsistent histories are indicated. Conceptual issues as to the interpretation of the different time operators are outlined with particular emphasis on the notion of temporal nonlocality.

1. INTRODUCTION

There is a long tradition of approaches toward a time observable which does not commute with all other observables, i.e., is not an element of the center of the algebra of observables of the system considered. This tradition is intimately connected with the history of the energy–time uncertainty relation (Jammer, 1974), which still represents an intriguing and provocative topic (see, e.g., Unruh and Wald 1989; Busch, 1990; Isham, 1993; Atmanspacher, 1994). A basic objection against a time operator T not commuting with a suitable energy operator, the (bounded or even discrete) Hamiltonian H of a system, was formulated by Pauli (1933a). If H and T are required to act in a Hilbert space of square-integrable functions, then Pauli's theorem says essentially that a self-adjoint time operator T_L with the entire real axis as its spectrum and with

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$$i[H, T] \neq 0 \tag{1}$$

does not exist.

Within recent years, two basic approaches have been developed to circumvent Pauli's objection. One of them refers to the generalization of quantum mechanical observables in terms of positive-operator-valued (POV) measures. Those measures are generally not self-adjoint, and therefore Pauli's theorem does not apply to them. Busch *et al.* (1994) and Giannitrappani (1997) showed how quantum mechanical time operators T_B that do not commute with the proper Hamiltonian of a system can be introduced. It is important, however, to stress that those time operators are not universally given, but depend on the context of the specific system.

Another route toward a noncommutative time operator was opened in the mid-1970s when Misra and others (Misra, 1978; Misra *et al.*, 1979; cf. Wightman, 1982) suggested the definition of a self-adjoint time operator T_L based on a projection-valued (PV) measure in the Liouville representation of quantum theory. As will be argued below, this representation can be interpreted as a metarepresenation with respect to the usual Hilbert space formulation, hence Pauli's theorem does not apply to the corresponding algebra of operators, often called superoperators. Since this algebra differs from that of operators in Hilbert space, T_L as a superoperator has a conceptual status different from T_B.

In view of this difference, it is interesting to explore whether the two types of time operators share common properties, what they mean in a given situation, and how they are related to each other. These questions form the central part of this paper. They will be addressed after a brief introduction to POV time T_B (Section 2) and time as a PV superoperator T_L in the Liouville representation (Section 3). The material in these two sections is kept as compact as possible; it is in no way intended to substitute for the more detailed literature. In Section 4, a way is discussed to understand a number of important aspects of the relationship between T_B and T_L. Section 5 discusses the kinds of temporal nonlocality implied by the two noncommutative time operators. Of special interest is their relationship to a temporal nonlocality that follows from temporal Bell inequalities as derived within the framework of consistent and inconsistent histories. Section 6 summarizes the main arguments.

Let us emphasize that the present article does not present novel rigorous results in a formal sense. Rather, it discusses a collection of approaches and arguments and their mutual relationships with respect to an issue that we consider to be one of the most significant issues of modern physics, its philosophy, and presumably even of other sciences. The issue is the existence of a time *observable* in addition to time as a *parameter*, together with its consequences.

2. POV TIME

In traditional quantum theory, observables are defined to formalize properties of a given system. This is usually done in such a way that the corresponding operators are chosen to be projection-valued self-adjoint operators. This reflects the assumption that any measurement of an ("elementary") observable essentially provides an outcome in the sense of a binary (yes/no) alternative [cf. von Neumann's (1932) expectation value postulate]. POV measures are concepts for observables that are more general than PV measures. They are, in general, not self-adjoint, i.e., they are not projectors. Their eigenfunctions (if they exist) are not necessarily orthogonal. As indicated by Davies (1976), Holevo (1982), Ludwig (1983), and others [most recently Braunstein *et al.* (1996) and Busch *et al.* (1996)], the concept of POV measures is particularly important for general measurement theory. For a more recent treatise describing other applications of POV measures see Busch *et al.* (1996).

A POV measure is a triple (Λ, Σ, E), where Σ is a ring (e.g., a σ-field) of subsets of a set Λ and E is an operator-valued set function on Σ with the following properties (Berberian, 1966):

- E is positive, that is, $E(M) \geq 0$ for each $M \in \Sigma$.
- E is additive, that is, $E(M \cup N) = E(M) + E(N)$ whenever M and N are disjoint subsets in Σ $(M \cap N = \emptyset)$.
- E is continuous in the sense that $E(M) = \sup E(M_n)$ whenever M_n is an increasing sequence of sets in Σ whose union is also in Σ.

[Another, more convenient formulation is to say that E is a POV measure on (Σ, Λ) if E is positive, additive, and continuous.]

A POV measure is a probability measure if it satisfies these conditions and is also normalized: $E(\Lambda) = 1$. A probability measure is a PV measure if it is multiplicative: $E(M \cap N) = E(M)E(N)$. In this case, the corresponding operator is idempotent and self-adjoint, and the set of its eigenfunctions (if they exist) is orthogonal. The characteristic function of a POV measure is the entire interval [0,1], whereas that of a PV measure is the set $\{0,1\}$. Hence, PV measures correspond to observables forming a Boolean algebra of measurements with binary alternatives (Primas, 1983), whereas POV measures correspond to a non-Boolean algebra. Sometimes, those measurements have been called "fuzzy," thereby suggesting some misleading relation to the theory of fuzzy sets. The theory of fuzzy sets (Klir and Folger, 1988) provides tools beyond conventional probability theory. While probabilities in the conventional sense are defined as probabilities of binary alternatives, fuzziness is a measure introduced at the level of the alternatives themselves, generally of infinitely many alternatives (Atmanspacher, 1989). POV observables are "unsharp" in the terminology of Busch *et al.* (1996). They are formally based within conventional probability theory, so we avoid the notion of fuzziness here.

Jauch and Piron (1967) used POV measures to generalize the notion of a localizable state of a system to which the entire body of traditional quantum mechanics refers although its logical structure is independent of localizability. A state of a system is localized if, given a domain D in regular position space, there is a state for which the probability for the system to be in D is one. Formally, this corresponds to the existence of three well-defined, commuting operators corresponding to the components of a position observable (Bacry 1988, Chapter 4). The position of a localizable system is always represented by a PV measure, say Y. Due to a theorem of Naimark (1943; see also Sz.-Nagy and Foias, 1970; Beals, 1977) for every POV measure X in a Hilbert space $\mathcal{H}$ an extension $\mathcal{H}^+$ of $\mathcal{H}$ and a PV measure Y in $\mathcal{H}^+$ can be constructed such that $X = PYP$, where P is a projection from $\mathcal{H}^+$ to $\mathcal{H}$ which does not commute with all the operators Y. Then the projections defined by $F = P \cap X$ represent a generalized spectral measure giving rise to a generalized concept of localizability. In this way, POV measures provide a framework in which the problem of the localizability of massless particles with spin [such as photons (Pauli, 1933b)] can be understood in terms of a generalized, “weak” localizability due to a suitable POV measure.

For the definition of a time operator based on a POV measure, one can proceed in the following manner. If H is the Hamiltonian of a system S, then $t \mapsto e^{-iHt}$, $t \in \mathbb{R}$, is a unitary representation V_t of the time translation group. If, furthermore, Θ is a time interval during which an event is expected to occur with some probability Trace(ρB) in a state ρ of S and for a suitable dynamical variable $B(\Theta)$, then $B(\Theta)$ satisfies.

$$V_t^* B(\Theta) V_t = B(\Theta - t) \tag{2}$$

where

$$V_t = \mathrm{e}^{-iHt}$$

Such a dynamical variable is a POV measure for a time observable (Busch *et al.*, 1994; Giannitrappani, 1997). Its construction is possible in specific cases, but cannot be universally prescribed; it depends crucially on contexts given by the system considered. On the basis of B, a time operator (briefly: POV time) can be defined according to

$$T_B = \int tB(dt) \tag{4}$$

which is (in general) *not* self-adjoint and fulfills the commutation relation

$$i[H, T_B] = \mathbb{1} \tag{5}$$

without contradicting Pauli's theorem (Busch *et al.*, 1994, Giannitrappani, 1997).

POV time operators T_B have no spectral decomposition in the usual sense of orthogonal eigenfunctions. Their definition presupposes a nonstationary dynamical observable B:

$$\frac{d\langle B\rangle}{dt} \neq 0.$$

For a stationary state, the temporal derivative of $\langle B\rangle$ vanishes. For instance, if $\langle B\rangle$ were taken as an energy of a stationary state, this would mean that energy is a "sharp" observable with vanishing bandwidth. In case of a nonstationary date, B gives rise to a so-called "unsharp" observable in the sense of an "unsharp" time of occurrence of an event. If $\langle B\rangle$ were an energy in this case, then there is a nonvanishing bandwidth ΔE, corresponding to a time interval $\Delta t = \hbar/\Delta E$ (Mandelstam and Tamm, 1945) within which no binary, (yes/no) alternatives are possible. At this point, the question remains open whether this "unsharpness" is simply of statistical significance or reflects a fundamental indeterminacy of individual quantum events.

3. PV TIME

3.1. Time Operator in Dynamical Systems

From a different point of view, Misra and others (Misra, 1978, Misra *et al.*, 1979a,b; see also Prigogine, 1980; Suchanecki, 1992) introduced a time operator T_L on the basis of the Liouville representation of a dynamical system. A related definition of a time operator, based on the theory of stochastic processes, was proposed by Tjøstheim (1976) and Gustafson and Misra (1976); see also Primas (1997a). In the following, we focus on the Liouville representation, in which the time operator is a shift operator T_L

$$U_t^* T_L U_t = T_L + t\,\mathbb{1} \tag{7}$$

with

$$U_t = e^{-iLt} \tag{8}$$

The time operator T_L does not commute with the Liouvillean L as defined through

$$L\rho = i\,\frac{\partial\rho}{\partial t} \tag{9}$$

Since the spectrum of L is not bounded, the commutation relation

$$i[L, T_L] = \mathbb{1} \tag{10}$$

does not contradict Pauli's theorem (see remarks below concerning the relevance of L as an energy operator). In contrast to T_B, the time operator T_L is self-adjoint and has a unique family $\{F(.)\}$ of spectral projections such that a spectral decomposition

$$T_L = \int t\, F(dt) \tag{11}$$

exists (more about this spectral decomposition later).

As Misra (1978) has shown, a necessary condition for the existence of T_L for a classical dynamical system is mixing, and a sufficient condition is that its Kolmogorov–Sinai entropy h_T (Kolmogorov, 1958; Sinai, 1959) is positive-definite, $h_T > 0$ (see Suchanecki, 1992, for more details.) In the framework of finite-dimensional nonlinear dynamical systems, this means that the system is intrinsically unstable in the sense of exponentially diverging trajectories (K-flow, chaos) due to positive Liapunov exponents. In the framework of stochastic processes, this is to say that the existence of a projection-valued operator T_L requires the dynamics of the system to be backward deterministic and forward nondeterministic (Primas, 1997a). In both frameworks, the K-flow condition is crucial.

The associated breaking of the time-reversal symmetry of the deterministic evolution due to L can be formally achieved if the unitary group U_t generated by L is mapped onto a semigroup W_t (more precisely: one of two possible semigroups) according to

$$W_t = \Lambda U_t \Lambda^{-1} \tag{12}$$

where Λ characterizes an invertible, nonunitary, positivity-preserving transformation $\rho \mapsto \tilde{\rho} = \Lambda\rho$ such that

$$\Lambda U_t \rho = W_t \Lambda \rho \tag{13}$$

The irreversible, stochastic semigroup W_t acts on the states $\tilde{\rho}$ by

$$\tilde{\rho}_t = W_t \tilde{\rho} \tag{14}$$

where $\tilde{\rho}$ can be interpreted as the state of an intrinsically random system, for instance belonging to the class of exact systems (Mackey, 1992; Suchanecki, 1992).

The transformation Λ can be constructed as

$$\Lambda = S^{1/2}, \qquad \Lambda^* \Lambda = S \tag{15}$$

if S is a Liapunov function of the system under study. Such a Liapunov function is defined by the conditions $S > 0$, $dS/dt \leq 0$ (or vice versa) which are typically satisfied for an entropy close to thermal equilibrium. [Misra (1978) excludes the equilibrium situation itself by requiring $S > 0$, $dS/dt < 0$.] In this sense, the dynamics according to W_t can be interpreted as an irreversible approach toward an equilibrium state.

It has been suggested (Atmanspacher and Scheingraber, 1987) that the restriction to the neighborhood of equilibrium situations be generalized by replacing the notion of entropy by a property defined information. This allows us to consider situations such as attractors of dissipative systems far from thermal equilibrium and other "dynamical" equilibria. A first step in this direction involves defining an information operator M according to [for details see Atmanspacher (1997) and references given there]

$$U_t^* M U_t = M - h_T t \mathbb{1} \tag{16}$$

again with

$$U_t = e^{-iLt} \tag{17}$$

The Kolmogorov–Sinai entropy h_T is an empirically accessible (Grassberger and Procaccia, 1983) dynamical invariant of the system. Clearly, M is a function of T_L, $M = M(T_L)$. It is important to realize that the definition of M is *more general* than that of T_L insofar as M can be defined even for nonmixing ergodic systems with $h_T = 0$, whereas a necessary condition for the existence of T_L is strong mixing, sometimes even $h_T > 0$. While T_L, if it exists, does not commute with the Liouvillean L, M commutes with L iff $h_T = 0$, and M does not commute with L iff $h_T > 0$ (Atmanspacher and Scheingraber, 1987; Atmanspacher, 1997):

$$i[L, M] = h_T \mathbb{1} \tag{18}$$

There are a number of conceptual problems with the approach originally proposed by Misra (1978). First of all, and most obvious, the operator T_L is introduced as a shift operator within the framework of a unitary evolution according to U_t, but finally it turns out that a necessary condition for its existence is a semigroup evolution, e.g., given by W_t. This problem is reflected by the fact that the Λ transformation relating U_t and W_t to each other breaks the time-reversal symmetry of the unitary evolution of ρ. Wightman's question as to the physical meaning of ρ (Wightman, 1982) hits precisely the same point. It may be speculated that the transformation between ρ and ρ can be regarded as a transformation between linear infinite-dimensional systems (stochastic processes, Wiener chaos) and nonlinear finite-dimensional systems (chaos à la Kolmogorov and Sinai). Some indications in that direction can be found in Braunss (1985) and Suchanecki (1992). Mackey (1992) interprets Λ as a transformation that basically creates a noninvertible "exact" system as a projection ("trace") of an invertible K-flow.

Another basic issue is the fact that both T_L and M are defined for classical systems so far. A quantum analog to the KS entropy in infinitely many dimensions was introduced by Connes *et al.* (1987), but it is not easy to interpret this analogy at a conceptual level. Also, more recent attempts to define a time operator for quantum systems in the framework of rigged Hilbert spaces remain to be evaluated in detail. In rigged Hilbert spaces the concept of self-adjointness can be rigorously applied to operators, which are then denoted "essentially" self-adjoint, meaning that their extension into usual Hilbert space is self-adjoint. Generalized spectral decompositions in rigged Hilbert spaces can lead to eigenvalues with nonvanishing imaginary part, e.g., for resonances (Böhm and Gadella, 1989).

3.2. Superoperators?

Misra *et al.* (1979b) have argued that it is necessary to proceed to the level of so-called superobservables and superoperators if entropy or information is to be formally incorporated as a Liapunov function into a consistent formal representation of quantum and classical systems. Today it is well known that the Liouville representation used by Misra *et al.* is a special case of the standard representation, a reducible representation of quantum theory (Haagerup, 1975), which makes it possible to deal with commutative ("classical") as well as noncommutative ("quantum") properties of a system in one and the same formal framework. It is not the goal of this paper to present this formal approach in detail; for a brief and illustrative introduction in this regard (however, without particular reference to entropy or time observables) see Grelland (1993).

From a conceptual point of view it is interesting to try to understand the distinction between ordinary quantum mechanics with its states and observables and the Liouville approach requiring superstates and superobservables. In fact, the Liouville approach reformulates the algebra of observables of ordinary quantum mechanics as a vector space whose elements are superstates on which superoperators act. In terms of algebraic quantum theory, this amounts to nothing else than a GNS (Gel'fand–Naimark–Segal) representation not involving the notion of superoperators. This has the consequence that, e.g., the expectation value of T_L is defined by

$$\begin{aligned} \langle T_L \rangle_\rho &= (\bar{\rho}, T_L \bar{\rho}) \\ &= \Phi \langle \bar{\rho}^* T_L \bar{\rho} \rangle \end{aligned} \tag{19}$$

where $\bar{\rho} = \rho - \rho_{eq}$ is an excess distribution function denoting the distance of ρ from the microcanonical equilibrium distribution ρ_{eq}, and Φ is a suitable reference state for the GNS construction, possibly ρ_{eq}. The expectation value of T_L is called the *internal time* of the distribution ρ, if $\bar{\rho}$ is an eigenfunction (eigenoperator) of T_L. The system's internal time advances as the time parameter t labeling the dynamical evolution U_t (or W_t) increases (Misra *et al.* 1979a; Lockhart *et al.*, 1982):

$$\langle T_L \rangle_{\rho_t} = \langle T_L \rangle_{\rho_0} + t \tag{20}$$

In more complicated cases, e.g., if $\bar{\rho}$ is not an eigenfunction of T_L, but a combination of eigenfunctions corresponding to two or more distinct eigenvalues, one can define an "average internal time" of ρ which behaves equivalently (Misra *et al.*, 1979a).

The "second order" way of thinking connected with the concept of superoperators can be illustrated by an information operator such as defined above. The eigenvalues of M refer to the amount of information with respect to a property of a system that can (under certain circumstances) be made available for an observer. The self-adjointness of M expresses a binary alternative with respect to the question of whether a binary alternative at the level of first-order properties has been decided. By contrast, ordinary operators refer to those properties themselves (positions, momenta, spins, energies, etc.). In this sense, an information operator represents an element of a "metadescription," referring to "metaproperties" of a system. It belongs to an algebra of "metaobservables" which are often denoted as superoperators. [Braunss (1985) indicated a generalized version of such an idea in terms of a hierarchy of algebras of observables and their associated state spaces. Primas (1963) introduced superoperators in the context of purely noncommutative systems (with irreducible factor type I algebras) without classical observables; see also references given there. Some more remarks about the relationship between metaobservables and observables in the framework of a standard representation will be made in Section 4.]

This is also the case for any other operator that is defined at the same level as M, in particular for T_L and the Liouvillean L. Hence, these operators play a role that differs from that of observables in ordinary ("first order") quantum theory. Since the Liouvillean can be written as the Poisson bracket of the Hamiltonian and an appropriate state function,

$$L\rho = \{H, \rho\} \tag{21}$$

its eigenvalues can be interpreted as differences of energies. However, two points must be emphasized in this context. First, this is not the most prominent meaning of the Liouville operator. Somewhat confusingly, L has often been used as an evolution operator for states as well as observables [see Antoniou and Suchanecki (1997) for a clarification], and there are Liouville-type equations that are more general than its prototype $L\rho = i\partial\rho/\partial t$. Second, it remains to be clarified in detail under which conditions L can be faithfully interpreted as an observable for energy differences. Recent work of Ban (1991) and Grelland (1993) gives first indications of possible approaches in this direction.

As to the time operator T_L, the situation is a bit more transparent. The eigenvalues of T_L are sometimes denoted in terms of "internal time," sometimes in terms of "age." Both concepts acquire significance through the time scale $1/h_T$ characterizing specific properties of the information flow of a system according to M (Shaw, 1981; Goldstein, 1981; Farmer, 1982; Atmanspacher and Scheingraber, 1987; Caves, 1994). For systems with $h_T = 0$, no such information flow exists, hence T_L cannot be defined for such systems. If $h_T > 0$, then the system is mixing and sensitive to initial conditions (in the sense of positive Liapunov exponents). The time scale given by the inverse of the information flow rate h_T is the time scale on which correlations typically decay. In other words, the inverse of h_T is a time scale for which the future behavior of the system can be reasonably well predicted, given the accuracy desired for the properties whose values are to be predicted. Therefore, T_L refers to a (second order) property of (first order) properties of a system in the same manner as M does [remember that $M = M(T_L)$, such that the logical level of M and T_L is the same]. Although the inverse of h_T is usually measured in terms of an ordinary parameter time, its significance reaches beyond such a concept.

4. RELATIONS BETWEEN T_B AND T_L

A most significant formal difference between T_B and T_L is given by the fact that POV time is in general not self-adjoint, whereas a time operator in the sense of Misra, i.e., in the Liouville representation, is a self-adjoint PV measure. As mentioned above, the self-adjointness of an operator is usually taken as a necessary condition for a description of Boolean measurements of the corresponding observable. More generally, this means that any measurement can be decomposed into more elementary measurements (projections), i.e., measurements with binary (yes/no) alternatives. With particular respect to T_L, this implies that the description of the history of a system cannot be decomposed into infinitesimally small subhistories.

For nonlinear systems evolving as a function of a continuous parameter time, the K-flow condition imposes restrictions on the temporal extension of those time intervals for which binary alternatives are possible. This is so because the definition of a K-flow comprises the definition of a partition on the relevant state space: the generating partition (Cornfeld *et al.* 1982; Crutchfield, 1983). Since the boundaries of phase cells of a generating partition are mapped onto themselves under the evolution of the system, any generating partition is a Markov partition and U_t can be transformed into a Markov process W_t. [This has been demonstrated for the discrete baker's transformation by Misra *et al.* (1979a).] Even continuous K-flows therefore contain an element of discreteness that is crucial for the definition of a self-adjoint T_L and its spectral decomposition. The inverse of the KS entropy h_T serves as a measure for the mean temporal distance between successive binary alternatives and may be understood as a system-specific "unit of aging." The "age" of a system is then given as the number of such *system-specific* time intervals rather than simply by a *universal* parameter time interval. [Note that this concept of "age" is at variance with that of Misra *et al.* (1979a,b), who use "age" and "internal time" synonymously.]

POV time, on the other hand, is independent of any discretization of the relevant phase space. If such a discretization is introduced by a proper partition, then POV time still refers to continuous trajectories within each individual discrete phase cell. Subhistories corresponding to an evolution within (the interior of) those phase cells are *not* objects of binary alternatives, since they cannot be described by projections. Such subhistories are "unsharp" (compare Antoniou and Suchanecki, 1997) in the sense that they are not further decomposable. Their temporal extension is given by the same time scale as the temporal distance between successive binary alternatives [compare Δt due to Mandelstam and Tamm (1945) as discussed in Section 2]. The notion of an internal time of a system reflects the parametrization of the evolution of a system within this time scale without any binary alternatives which would allow us to follow this evolution empirically. The notion of a "hidden history" (see next section) may be used for such a situation. It offers an interesting analogy to the problem of measurement, if one considers such a "hidden history" in terms of a (stochastic) evolution of a nonempirical (ontic) trajectory over a time interval corresponding to the (average) size of a phase cell. Empirical (epistemic) binary alternatives and associated self-adjoint projections are possible right after that time interval. In this sense, a generalized "process of measurement" can be conceived as the projection of a "hidden history" onto a binary alternative, and a generalized "reduction of a state" takes place within each phase cell.

From a similar perspective one can understand Misra's (1995) recent result that the purely continuous Klein–Gordon evolution of massive particles allows self-adjoint extensions T_L of a time operator only for discrete subgroups $U_{n\tau}$ of U_t (with $\tau = \hbar/mc^2$). The restriction to discrete subgroups represents nothing else than the introduction of a proper partition. It implies that elementary propositions corresponding to binary alternatives are meaningful only with respect to time intervals τ depending on the mass of the particle considered. Within those intervals, the "dynamics" is not defined within a Boolean algebra of binary alternatives (Atmanspacher, 1989). The whole discussion about "Zitterbewegung" and other kinds of vacuum fluctuations refers to such a situation. Misra is entirely right with his comment that such phenomena are epistemically unaccessible in the sense of binary alternatives. Of course, this does not rule out that they may be ontically relevant and have consequences that are epistemically accessible.

As repeatedly mentioned above, another major difference between T_B and T_L consists in the fact that T_B is an element of an algebra of (first order) observables for the system considered, whereas T_L is an element of an algebra of (second order) metaobservables, referring to properties of observables rather than observables themselves. This can be illustrated most clearly if the meaning of operators such as T_L or M is expressed in terms of correlations, information, or entropy. Such variables are introduced as Liapunov functions and imply a broken time symmetry. This is an essential point concerning the problem as to whether the corresponding metaobservables can *equivalently* be understood as ordinary (first order) elements of a reducible W*-algebra in the framework of a standard representation, obtained via GNS construction. The concept of a metaobservable does *explicitly* refer to time-reversal symmetry breaking, whereas observables in the standard representation do not presuppose such a symmetry breaking in such an obvious way (one may speculate that it can be provided by a proper GNS construction).

It is also worth addressing the emergence of classical observables in the framework of a metadescription. Lockhart and Misra (1986; cf. Primas, 1987) demonstrated that a generalized understanding of measurement (not restricted to the measurement problem of quantum mechanics) can be achieved if the measurement process is considered in terms of a "metadynamics" in the sense of a K-flow. (Again, let me mention that this description may be put into the formal framework of a standard representation, but this is not the issue here.) By contrast to Hepp's (1972) algebraic approach toward measurement, which depends on the limit of $t \rightarrow \infty$ to generate irreversible facts in the sense of classical observables and structures, the Lockhart and Misra scenario describes a "gradual" kind of irreversibility that does not allow an "unfinished" measurement (i.e., for any finite time) to be undone (cf. Primas, 1997a). Formally, this suggests something like an increasing number of elements in the center of the algebra of observables. It may be interesting to speculate that this formal point of view corresponds to an irreversible process at the level of a metadescription. At such a level, the first-order algebra of observables becomes a second-order vector space in which a dynamics according to M with characteristic time scales according to h_T might provide a description of an increasing center of a first-order algebra of observables.

The crucial step to relate T_B and T_L to one another is to identify a Liapunov function as the dynamical variable B. Misra (1978) used an entropy to do so, and Atmanspacher and Scheingraber (1987) effectively generalized this to (syntactic) information. Both Liapunov functions, entropy and information, require that time-reversal symmetry be broken. If the character of entropy or information as a *second-order* observable is not explicitly kept in mind, it is natural that such proposals lead to considerable confusion, since entropy and information are not elements of "ordinary" algebras of observables. If one wants to stay at the level of first-order dynamical variables, then one might consider quantities corresponding to the energy flow through an open system, such as an energy loss which could be incorporated into a more familiar formulation. In a recent paper, Primas (1997b) proposed such a procedure, following ideas of Meixner (1961). Like any other characterization of irreversibility, this also requires that the time-reversal symmetry of a unitary evolution has to be broken.

Is it possible to relate the time-reversal symmetry breaking with the extension of a (first order) POV measure T_B to a self-adjoint PV measure T_L? Busch *et al.* (1994) introduced POV time as time-translation covariant, so the symmetry breaking has to happen together with the extension into self-adjointness *and* together with the step from first-order to second-order observables, including the selection of a Liapunov function. As a formal expression of this somewhat sophisticated relationship between T_B and T_L, Antoniou (1997) proposed the conjecture that $T_L = [T_B, .]$ (analogous to $L = [H, .]$), at least in special cases.

Moreover, it might be interesting to work out Giannitrappani's (1997) remark that POV time has two disjoint families of (generalized) eigenvectors. They might be discussed according to two different semigroup evolutions, which should be uniquely constructable by Naimark's theorem [Naimark (1943); cf. Jauch and Piron (1967) and the discussion in Section 3; for a conceptually similar, but formally different approach see Lockhart *et al.* (1982) and Antoniou and Prigogine (1993)]. It has to be pointed out, though, that the *formal* decomposition of a unitary evolution of a closed system into two semigroups does not already imply that the *physical* dynamics of the same closed system is irreversible in the sense that facts are generated (Primas, 1997a). Assuming that the two semigroups can be interpreted in terms of different temporal directions, the question would be left open of how and why one of those directions/semigroups seems to be favored by irreversible processes as realized in the material world, once the system is an open system.

5. TEMPORAL NONLOCALITY

There is an interesting correspondence between the preceding discussion of time observables and an approach that has become popular under the name of "consistent histories." This approach was originally introduced by Griffiths (1984). Omnès (1992) used it as a central feature in his interpretation of quantum theory, and there are certain links to the concept of decoherence (Gell-Mann and Hartle, 1990; Zurek, 1991; Kiefer, 1996) that gave rise to the notion of "decoherent histories," sometimes used instead of "consistent histories."

The formal definition of a history is given in terms of a time-ordered sequence of commuting projectors $P^k_{\alpha_k}(t_k)$,

$$[P_\alpha] = P^1_{\alpha_1}(t_1)P^2_{\alpha_2}(t_2) \cdots P^n_{\alpha_n}(t_n) \tag{22}$$

which are mutually exclusive,

$$P^k_{\alpha_k} P^k_{\beta_k} = \delta_{\alpha\beta}\, P^k_{\alpha_k} \tag{23}$$

and exhaustive binary (yes/no) alternatives, labeled by α,

$$\sum_{\alpha_n} P^k_{\alpha_k} = 1 \tag{24}$$

Sequences such as $[P_\alpha]$ provide descriptions of a set of possible histories with respect to a coarse graining introduced by a suitable partition. The coarsest history consists of no projections at all, but just the unit operator. A completely fine-grained history (such as in Feynman's sum-over-histories framework) would be specified by the values of a complete set of operators at all times and therefore would have to be based on an infinitely refined partition. Since such a history cannot consistently be assigned probabilities, *consistent* histories require suitable *finite* partitions to be introduced. Thus the task is to find those partitions that provide a consistent attribution of probabilities (e.g., satisfying probability sum rules).

There is a lot more to say about the application of the concept of decoherence to the histories approach as, e.g., developed by Gell-Mann and Hartle (1990), Zurek (1991), and others. This is not the place to discuss all this in detail. A basic point of criticism seems to be that the formalism partly refers to isolated systems, partly to open systems, and this entails severe conceptual problems. Stripped of the corresponding confusion, the idea of decoherent histories basically resembles an exponential divergence of trajectories such as in K-flows, and the binary alternatives associated with the projectors $P^k_{\alpha_k}(t_k)$ are simply due to the membership function with respect to individual phase cells of the relevant partition. This is to say that each projection refers to an element of the discretized history $[P_\alpha]$. In the light of the arguments of Section 3, Zurek's recent modification of a "predictability sieve" in his decoherence approach goes in the same direction (Zurek, 1994). [The claim that decoherence provides a description of measurement can only be relevant if the process of measurement is conceptually disentangled from the generation of classical observables. Classical observables require disjoint states, which decoherence does not produce *in general*. There is, however, no apparent reason why measurement in the general sense of any interaction between an object and its (quantum) environment (Griffiths, 1994), not restricted to laboratory situations, might not be reasonably described in terms of decoherent processes.]

Rather than elaborating further on these details here, let us focus on another feature of the histories approach, namely on *inconsistent* histories. This term describes the impossibility of assigning probabilities to completely fine-grained histories in a consistent way. Based on a proposal by Leggett and Garg (1985), Paz and Mahler (1993; see also Mahler, 1994) studied this inconsistency in terms of a temporal analog of Bell's inequalities, called temporal Bell inequalities. The temporal character of the inequalities implies that their violation rejects hidden histories rather than hidden variables. Paz and Mahler's numerical results for a specific quantum transition network model show that such a system indeed violates temporal Bell inequalities in analogy to Aspect's experimental results for ordinary Bell inequalities. Corresponding experiments have not yet been carried out.

In the framework of the histories approach, a violation of temporal Bell inequalities indicates the inconsistency of histories that are too fine-grained. Now, of course, the question is: what is a good criterion for a proper coarse graining or, in other words, for a proper partition? Paz and Mahler's criterion is a time-scale argument derived from the incoherent part of the evolution of the reduced density matrix for the system they considered. The corresponding time scale τ_c is basically a relaxation time, which is infinitely long for closed systems with purely coherent evolution and vanishes for strictly random processes. Since the incoherent part of the evolution is responsible for any nonvanishing information flow in the system, an interpretation in terms of an inverse information flow rate (such as the KS-entropy) suggests itself.

Any sensible interpretation of the "evolution" of a system within τ_c in terms of hidden histories is ruled out by the violation of temporal Bell inequalities. This is to say that within τ_c there is no history (more precisely: no consistent history in terms of a sequence of events) that can be determined by successive binary alternatives. As in the case of the generating partition, even continuous evolutions appear discretized if they are to be described by sequences of such alternatives, i.e., by projectors. It is exactly in this sense that one can speak of some kind of *temporal nonlocality*, a notion motivated by comparison to the fundamental concept of nonlocality that is commonly accepted in present-day quantum theory. The similarity between a temporal nonlocality due to inconsistent histories and a temporal nonlocality due to the time operator T_B is striking since it not only offers a number of almost identical crucial features (cf. Section 3), but also gives a very basic motivation for temporal nonlocality due to the noncommutativity of T_B. This is related to temporal nonlocality due to T_L insofar as the existence of a dynamical variable B for a nonstationary state is a necessary condition for T_L as well (see also Misra and Prigogine, 1983; Martinez and Tirapegui, 1985; Suchanecki *et al.*, 1994; Atmanspacher, 1997).

These analogies notwithstanding, it has to be emphasized that any temporal nonlocality due to T_L is conceptually different from fundamental quantum-nonlocality in several essential regards: (1) T_L is a *second-order* observable involving the concept of a Liapunov function, (2) the commutator of M and L is, by contrast to Planck's action $\hbar$, not a universal constant, but the system-specific KS-entropy h_T, and (3) the definition of T_L as well as M presupposes the selection of a direction of time, i.e., is based on the irreversible evolution of the distribution function $\tilde{\rho}$. Therefore, it is an ambitious goal to identify a time operator that is logically prior to T_L. A natural candidate might be T_B. POV time does not require a direction of time to be selected, although, if Giannitrappani's argument for two disjoint families of eigenfunctions of T_B holds, it entails an explicit option for time-reversal symmetry breaking. Hence, there is some appeal in considering POV time as a noncommutative time operator that is more basic than T_L. The concrete construction of T_L by the particular choice of a Liapunov function as a dynamical variable B for nonstationary states supports this suggestion.

Does POV time imply a temporal nonlocality as basic as the fundamental nonlocality of quantum theory [see Redhead (1987) for a detailed account of nonlocality from the perspective of the philosophy of physics]? In fundamental quantum nonlocality, the notion of subsystems of a system becomes, strictly speaking, inadmissible. For instance, it does not make sense to talk about two spatially separated photons in Aspect's experiment, as long as no interaction or measurement of the photon pair as a whole with some environment has taken place. [Recall that there are additional problems with respect to the localizability of photons, as mentioned in Section 2 (Jauch and Piron, 1967; Bacry, 1988)]. There is simply no spatial order in the sense of a distance between any particular locations within the system as a whole.

With respect to POV time nonlocality, the situation is different. The (formal, not necessarily physical) symmetry breaking of a unitary evolution in two different temporal directions according to two disjoint semigroups implies that temporal *order* in the sense of a distance between any individual instants must be well defined. This is a necessary but not sufficient condition for temporal *direction* in the sense of irreversibility. Genuine *atemporality,* something that is logically prior even to reversible evolution, is not addressed by POV time nonlocality. Needless to say, it is even less addressed by temporal nonlocality due to T_L, since T_L presupposes temporal direction in addition to temporal order.

6. SUMMARY

Two different ways to introduce time operators that are not commutative have been discussed and compared with each other. One of them, T_B (Busch *et al.*, 1994, Giannitrappani, 1997), is based on positive-operator-valued (POV) measures, not referring to elementary measurements in the sense of binary (yes/no) alternatives or—formally speaking—projectors. T_B is in general not self-adjoint and its set of eigenvectors is in general not orthogonal. Such a time operator does not commute with the Hamiltonian of the system considered. It provides a so-called "unsharp" characterization of the time of occurrence of an event within a given temporal interval.

Another noncommutative time operator, T_L (Misra, 1978), is based on projection-valued (PV) measures in the Liouville representation. It refers to binary alternatives corresponding to projectors and is self-adjoint with an orthogonal set of eigenvectors. Such a time operator does not commute with the Liouvillean of the system considered. It characterizes the "age" of a system, given in nondecomposable, system-specific units of the inverse of the KS entropy h_T.

There are a number of basic differences between T_B and T_L. First, T_L is defined at a second-order level where one deals with properties of properties (observables of observables) of a system. By contrast, T_B is defined at the (usually considered) first-order level and refers to properties (observables) of a system themselves. Thus, the two time operators belong to two different algebras of observables. Disregarding this difference inevitably leads to misleading interpretations and all sorts of corresponding confusion.

Second, T_B is independent of the selection of a temporal direction, whereas T_L presupposes such a selection in addition to an explicit breaking of time-reversal symmetry. This becomes obvious if one looks at the way in which a self-adjoint extension of T_B leading to T_L can be constructed. This is done by introducing a Liapunov function such as entropy or, more generally, information. Thus, T_B can be defined for reversible processes, whereas T_L is based on irreversibility. If the Liapunov function required for T_L is a second-order observable (which is the case for entropy and information), then the self-adjoint extension of T_B entails both the transition to irreversibility and the transition to a second-order level description.

A third major difference between the two types of noncommutative time operators can be recognized in the way they are related to temporal nonlocality. T_L, or some associated information operator M, refers to a time interval which is given by the relevant partition, and for which binary alternatives are possible (with respect to a given accuracy). This time interval is the inverse of the KS-entropy h_T, a dynamical invariant of the system that is intimately related to its generating partition.

POV time T_B refers to an unsharp time of occurrence of an event rather than an "age." This expresses a temporal nonlocality in the sense that within a time interval with nonvanishing and finite size there is no way to assign a temporal instant to an event. One may speculate that the second-order argument concerning binary alternatives with respect to "aging" in units of $1/h_T$ is nothing else than an expression of first-order "unsharpness" of events within $1/h_T$ from a different perspective. The essence of both viewpoints is temporal nonlocality within $1/h_T$.

Both perspectives, that of T_L and of T_B, can be conceptually related to each other within the histories approach (Griffiths, 1984) in the form developed by Gell-Mann and Hartle (1990). The notion of a consistent history is bound to a partition that cannot be infinitely refined, but has to be properly coarse grained. This is closely related to the significance of T_L. Inconsistent histories arise within a system-specific time scale corresponding to the (average) size of phase cells generated by a (proper) coarse graining. As Paz and Mahler (1993) have shown, inconsistent histories correspond to violations of temporal Bell inequalities, thus ruling out hidden histories in analogy to local hidden variables. The interpretation of T_B reflects this kind of nonlocality.

Temporal nonlocality in the two perspectives addressed by T_L and T_B must not be considered at the same logical level as fundamental quantum nonlocality. As far as we can see at present, fundamental quantum nonlocality requires the concepts of spatial and temporal order in their commonsense meaning to be given up entirely. This is neither the case for T_L nor for T_B. Some preliminary characterizations of temporal nonlocality in the context of the two time operators have been indicated. For an improved understanding it will be necessary to look for empirical consequences, to be proposed in another paper.

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